

Approximation capability of the convolution methods for fuzzy numbers

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Abstract

This paper shows that how to approximate general fuzzy number by using convolution method.

Key words: Fuzzy numbers; Convolution; Supremum metric; Differentiable

1 Instructions

The approximation of fuzzy numbers attract many peoples attentions [1,2,5,6]. Since differentiable fuzzy numbers play an important role in the implementation of fuzzy intelligent systems and their applications (see [3,4]). Chalco-Cano et al. [5,6] introduced a method based on the convolution to approximate non-differentiable fuzzy numbers by differentiable fuzzy numbers.

A significant advantage of this method is that it can generate differentiable fuzzy number such that the distance between which and the original fuzzy number is less than or equal to arbitrary predetermine positive number.

However, in the previous work, not all the fuzzy numbers can be approximated by this method. So it is natural for us to consider the question that can we use the convolution method to approximate general fuzzy numbers?

In this paper, we want to answer this question.

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2 Preliminaries

2.1 Fuzzy numbers

Let \mathbb{N} be the set of all natural numbers, \mathbb{R} be the set of all real numbers. For details, we refer the reader to references [7,8].

A fuzzy subsets u on \mathbb{R} can be seen as a mapping from \mathbb{R} to $[0,1]$. For $\alpha \in (0, 1]$, let $[u]_\alpha$ denote the α -cut of u ; i.e., $[u]_\alpha \equiv \{x \in \mathbb{R} : u(x) \geq \alpha\}$ and $[u]_0$ denotes $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$. We call u a fuzzy number if u has the following properties:

- (i) $[u]_1 \neq \emptyset$; and
- (ii) $[u]_\alpha = [u^-(\alpha), u^+(\alpha)]$ are compact intervals of \mathbb{R} for all $\alpha \in [0, 1]$.

The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$. In [5], a fuzzy number is also called a fuzzy number.

The following is a widely used representation theorem of fuzzy numbers.

Proposition 2.1 (*Goetschel and Voxman [10]*) *Given $u \in \mathcal{F}(\mathbb{R})$, then*

- (i) $u^-(\cdot)$ is a left-continuous nondecreasing bounded function on $(0, 1]$;
- (ii) $u^+(\cdot)$ is a left-continuous nonincreasing bounded function on $(0, 1]$;
- (iii) $u^-(\cdot)$ and $u^+(\cdot)$ are right continuous at $\alpha = 0$;
- (iv) $u^-(1) \leq u^+(1)$.

Moreover, if the pair of functions $a(\lambda)$ and $b(\lambda)$ satisfy conditions (i) through (iv), then there exists a unique $u \in \mathcal{F}(\mathbb{R})$ such that $[u]_\alpha = [a(\lambda), b(\lambda)]$ for each $\alpha \in (0, 1]$.

The algebraic operations on $\mathcal{F}(\mathbb{R})$ are defined as follows: given $u, v \in \mathcal{F}(\mathbb{R})$, $\alpha \in [0, 1]$,

$$\begin{aligned} [u + v]_\alpha &= [u]_\alpha + [v]_\alpha = [u^-(\alpha) + v^-(\alpha), u^+(\alpha) + v^+(\alpha)], \\ [u - v]_\alpha &= [u]_\alpha - [v]_\alpha = [u^-(\alpha) - v^+(\alpha), u^+(\alpha) - v^-(\alpha)], \\ [u \cdot v]_\alpha &= [u]_\alpha \cdot [v]_\alpha = [\min\{xy : x \in [u]_\alpha, y \in [v]_\alpha\}, \max\{xy : x \in [u]_\alpha, y \in [v]_\alpha\}]. \end{aligned}$$

The supremum metric on $\mathcal{F}(\mathbb{R})$ is defined by

$$d_\infty(u, v) = \sup_{\alpha \in [0, 1]} \max\{|u^-(\alpha) - v^-(\alpha)|, |u^+(\alpha) - v^+(\alpha)|\},$$

where $u, v \in \mathcal{F}(\mathbb{R})$.

2.2 Convolution of fuzzy numbers

In this paper, we want to discuss the properties of sup-min convolution $u \nabla v$ of fuzzy numbers u and v , which is defined by

$$(u \nabla v)(x) = \sup_{y \in \mathbb{R}} \{u(y) \wedge v(x - y)\}.$$

Remark 2.2 *In fact $u \nabla v = u + v$ for all $u, v \in \mathcal{F}(\mathbb{R})$. For details, see [7, 8].*

In the following, we list some symbols which are used to denote subsets of $\mathcal{F}(\mathbb{R})$.

- $\mathcal{F}_T(\mathbb{R})$ is denoted the family of all fuzzy numbers u such that u is strictly increasing on $[u^-(0), u^-(1)]$, strictly decreasing on $[u^+(1), u^+(0)]$, and differentiable on $(u^-(0), u^-(1)) \cup (u^+(1), u^+(0))$.
- $\mathcal{F}_N(\mathbb{R})$ is denoted the family of all fuzzy numbers u such that u is differentiable on $(u^-(0), u^-(1)) \cup (u^+(1), u^+(0))$.
- $\mathcal{F}_C(\mathbb{R})$ is denoted the family of all continuous fuzzy numbers, i.e., the family of all fuzzy numbers u such that $u : \mathbb{R} \rightarrow [0, 1]$ is continuous on $(u^-(0), u^+(0))$.
- $\mathcal{F}_D(\mathbb{R})$ is denoted the family of all differentiable fuzzy numbers, i.e., the family of all fuzzy numbers u such that $u : \mathbb{R} \rightarrow [0, 1]$ is differentiable on $(u^-(0), u^+(0))$.

Given a fuzzy number u in $\mathcal{F}_N(\mathbb{R})$, u need not be strictly increasing on $(u^-(0), u^-(1))$ and strictly decreasing on $(u^+(1), u^+(0))$. So

$$\mathcal{F}_T(\mathbb{R}) \subsetneq \mathcal{F}_N(\mathbb{R}).$$

Observe that p is differentiable on $(p^-(1), p^+(1))$ for all $p \in \mathcal{F}(\mathbb{R})$. Thus we know that, for each $u \in \mathcal{F}_N(\mathbb{R})$, the possible non-differentiable points are $u^-(1)$ and $u^+(1)$.

It is easy to check that

$$\mathcal{F}_D(\mathbb{R}) \subsetneq \mathcal{F}_C(\mathbb{R}) \cap \mathcal{F}_N(\mathbb{R}).$$

Chalco-Cano et al. [5] constructed fuzzy numbers w_p , $p > 0$, which are defined by

$$w_p(x) = \begin{cases} 1 - \left(\frac{x}{p}\right)^2, & \text{if } x \in [-p, p], \\ 0, & \text{if } x \notin [-p, p]. \end{cases} \quad (1)$$

Obviously, $w_p \in \mathcal{F}_D(\mathbb{R})$ for all $p > 0$. They presented the following result.

Proposition 2.3 [5] *If $u \in \mathcal{F}_T(\mathbb{R})$, then $u \nabla w_p \in \mathcal{F}_D(\mathbb{R})$.*

Notice that $d_\infty(u, u \nabla w_p) \rightarrow 0$ as $p \rightarrow 0$. Thus Proposition 2.3 indicates that every fuzzy number in $\mathcal{F}_T(\mathbb{R})$ can be approximated by fuzzy numbers sequences in $\mathcal{F}_D(\mathbb{R})$.

We can see that the fuzzy numbers w_p , $p > 0$, work as smoothers, which transfer each fuzzy number u to a differentiable (smooth) fuzzy number $u \nabla w_p$. This sequence of smooth fuzzy numbers construct a approximation sequence of the original fuzzy number u , i.e., $u \nabla w_p \rightarrow u$ as $p \rightarrow 0$.

Chalco-Cano et al. [6] further putted forward a method to define smoothers. Suppose that $p > 0$ is a real number and that $f : [0, 1] \rightarrow [0, 1]$ is a continuous and strictly decreasing function with $f(0) = 1, f(1) = 0$. A class of fuzzy numbers Z_p^f is defined by

$$Z_p^f(x) = \begin{cases} f^{-1}(\|x\|/p), & \|x\| \leq p, \\ 0, & \|x\| > p. \end{cases}$$

It is easy to show that $Z_p^f = w_p$ when $f = \sqrt{1-t}$. They gave the following result.

Proposition 2.4 [6] *If f is differentiable and $\lim_{\alpha \rightarrow 1-} f'(\alpha) = -\infty$, then $u \nabla Z_p^f \in \mathcal{F}_D(\mathbb{R})$ for each $u \in \mathcal{F}_T(\mathbb{R})$.*

Notice that $d_\infty(u \nabla Z_p^f, u) \rightarrow 0$ as $p \rightarrow 0$. This means that given f satisfies the above conditions, we obtain a smooth approximation $\{u \nabla Z_p^f : p > 0\}$ of the fuzzy number u . Different f corresponds to different sequence of smooth approximation.

3 Approximation

In the previous work, only fuzzy numbers in $\mathcal{F}_T(\mathbb{R})$ can be approximated. This type of fuzzy numbers have at most two possible non-differentiable points: the endpoints of the 1-cut. Whereas, an arbitrarily given fuzzy number may have other non-differentiable points or non-continuous points. So it is natural for us to consider the question that can we use the convolution method to approximate general fuzzy numbers?

In this section, to answer this question, it discusses how to choose smoothers to smooth an arbitrarily given fuzzy number. The following theorems are devoted to this problem.

Theorem 3.1 Suppose that $u \in \mathcal{F}_N(\mathbb{R}) \cap \mathcal{F}_C(\mathbb{R})$ and that $w \in \mathcal{F}_D(\mathbb{R})$, then $u \nabla w \in \mathcal{F}_D(\mathbb{R})$ when w satisfies the following conditions.

- (i) $w(w^-(0)) = u(u^-(0))$ and $w(w^+(0)) = u(u^+(0))$.
- (ii) If $u^-(1)$ is the inner point of $[u]_0$, then $w'_-(w^-(1)) = 0$.
- (iii) If $u^+(1)$ is the inner point of $[u]_0$, then $w'_+(w^+(1)) = 0$.

Theorem 3.2 Suppose that $u \in \mathcal{F}_C(\mathbb{R})$ and that $w \in \mathcal{F}_D(\mathbb{R})$, then $u \nabla w \in \mathcal{F}_D(\mathbb{R})$ when w satisfies the conditions (i), (ii), and (iii) in Theorem 3.1 and the following condition (iv).

- (iv) Given $x \in (u^-(0), u^+(0))$ with $u(x) = \alpha < 1$, if x is a non-differentiable point of u , then
 - (iv) (1) if $x < u^-(1)$, then $w'(w^-(\alpha)) = 0$;
 - (iv) (2) if $x > u^+(1)$, then $w'(w^+(\alpha)) = 0$.

Theorem 3.3 Suppose that $u \in \mathcal{F}(\mathbb{R})$ and that $w \in \mathcal{F}_D(\mathbb{R})$, then $u \nabla w \in \mathcal{F}_D(\mathbb{R})$ when w satisfies the conditions (i), (ii), and (iii) in Theorem 3.1, the condition (iv) in Theorem 3.2, and the following condition (v).

- (v) Given $x \in (u^-(0), u^+(0))$ with $u(x) = \alpha < 1$, if x is a non-continuous point of u , then
 - (v) (1) if $x < u^-(1)$, then $w'(w^-(\beta)) = 0$, where $\beta = \lim_{y \rightarrow x^-} u(y)$;
 - (v) (2) if $x > u^+(1)$, then $w'(w^+(\gamma)) = 0$, where $\gamma = \lim_{z \rightarrow x^+} u(z)$.

By above theorems, given an arbitrary fuzzy number, if the number of non-continuous points of u in $[u]_0$ is finite, then we can construct a sequence of smoothers v_n , $n \in \mathbb{N}$, such that $u \nabla v_n$ is a differentiable fuzzy number for all n , and that $\{u \nabla v_n\}$ converges to u in the supremum metric d_∞ .

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